EQUILIBRIUM EXISTENCE THEOREMS OF ABSTRACT ECONOMICS IN $H$-SPACES*

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In this paper, we prove a fixed point theorem in the product space of $H$-spaces. By applying our fixed point theorem, some equilibrium existence theorem of abstract economies are obtained under noncompact and nonconvex setting.

1. INTRODUCTION

In recent years, many authors have proved the equilibrium existence theorems of abstract economies with infinite dimensional strategy spaces and infinite number of agents, e.g. see Yannelis-Prabhakar, Toussaint, Tulcea, Tarafdar, Ding-Tan, Ding-Kim-Tan, Tian and Im-Kim-Rim. The most of the theorems generalize the basically important results of Shafer-Sonnenschein and Borglin-Keiding. To my best knowledge, all existence theorems mentioned above are proved by assuming that the strategy sets are convex or compact convex subsets of topological vector spaces. The assumptions are very restricted since the strategy sets of agents generally are not compact and convex in any topology of commodity spaces which may not have the linear structure and various kinds of preference and constraint correspondences will be encountered in generally economic situations. Thus, it is important and interesting to establish some equilibrium existence theorems of abstract economies with noncompact and nonconvex strategy sets of agents.

In this paper, we first prove a fixed point theorem in noncompact product $H$-spaces which extend Theorem 2 of Ding-Kim-Tan to $H$-space and is closely related to Theorem 2.1 of Tarafdar. Next, by applying our fixed point theorem, some equilibrium existence theorems for abstract economies with noncompact and nonconvex strategy sets in $H$-spaces are obtained which are either closely related to or generalizations of the corresponding results of Borglin-Keiding, Shafer-Sonnenschein, Tarafdar, Tian, Ding-Kim-Tan. For fixed points of upper semicontinuous contractible valued maps and its applications, see McLennan and McClendon.

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2. Preliminaries

Let $A$ be a subset of a topological space $X$. We shall denote by $2^A$ the family of all subsets of $X$, by $\text{int}_X(A)$ the interior of $A$ in $X$ and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of $A$.

The following notions were introduced by Horvath\textsuperscript{10, 11} and Bardaro-Ceppitelli\textsuperscript{1}.

A pair $(X, \{\Gamma_A\})$ is said to be an $H$-space (c-space according to Horvath\textsuperscript{11}) if $X$ is a topological space and $\{\Gamma_A\}$ is a given family of contractible subsets $\Gamma_A$ of $X$, indexed by $A \in \mathcal{F}(X)$, such that $A \subseteq A'$ implies $\Gamma_A \subseteq \Gamma_{A'}$. A nonempty subset $D$ of an $H$-space $(X, \{\Gamma_A\})$ is called $H$-convex if $\Gamma_A \subseteq D$ for each $A \in \mathcal{F}(D)$. An $H$-space $(X, \{\Gamma_A\})$ is said to be an l.c.-space if $X$ is a uniform topological space and if there exists a basis $(V_i)_{i \in I}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X : E \cap V_i(x) = \emptyset\}$ is $H$-convex whenever $E$ is $H$-convex where $V_i(x) = \{y \in X : (x, y) \in V_i\}$.

**Example 1** — Let $(Y, \mathcal{I}(D))$ is a gauge space (see, Dugundji\textsuperscript{8} p.199) where $D = \{d_i : i \in I\}$ is a separating family of gauges on $Y$. Let $\alpha = [0, 1] \times Y \times Y \to Y$ be a function such that

$$\alpha(0, x, y) = \alpha(1, y, x) = x, \text{ for all } (x, y) \in Y \times Y.$$ 

$A \subseteq Y$ is an $\alpha$-set if $\alpha([0, 1] \times A \times A) \subseteq A$. Let $\Gamma_A = \cap \{B : A \subseteq B \Rightarrow B \text{ is an } \alpha\text{-set}\}$. Assume that for any $x* \in Y$, the function $(x, t) \mapsto \alpha(t, x, x*)$ is continuous on $[0, 1] \times Y$ and for each $i \in I$ and for any $(t, x_i, y_i), (t, y_i, y_2) \in [0, 1] \times Y \times Y$,

$$d_i(\alpha(t, x_1, x_2), \alpha(t, y_1, y_2)) \leq \max \{d_i(x_1, y_1), d_i(x_2, y_2)\}.$$

Then $Y$ becomes a l.c.-space with $\Gamma_{\{y\}} = \{y\}$. If $D$ consists of one gauge $d$ alone, then $(Y, \mathcal{I}(d))$ is a metric l.c.-space. Hence, it is clear that each metric l.c.-space is an l.c.-space and the converse is false.

**Example 2** — Let $X$ be a locally convex Hausdorff topological vector space. For each $A \in \mathcal{F}(X)$, let $\Gamma_A = \text{co}(A)$, then $(X, \{\Gamma_A\})$ is an $H$-space. Since $X$ is locally convex, then the topology of $X$ can be deduced by a family $\mathcal{P} = \{p_i : i \in I\}$ of semi-norms $p_i$ on $X$. By the Example 2 of Dugundji\textsuperscript{8} (p.201), the family of sets $V_{p_i, \epsilon} = \{ (x, y) \in X \times X : p_i(x - y) < \epsilon \}$ for all $i \in I$ and $\epsilon > 0$ is a uniformity in $X$ and hence $X$ is a uniform space. For any convex subset $E$ of $X$, any $i \in I$ and any $\epsilon > 0$, we prove the set $B = \{x \in X : E \cap V_{p_i, \epsilon}(x) = \emptyset\}$ is $H$-convex. Let $A = \{x_1, \ldots, x_n\}$ is any finite subset of $B$, then there exist $y_1, \ldots, y_n \in E$ such that $p_i(x_k - y_k) < \epsilon$ for $k = 1, \ldots, n$. For each $z \in \text{co}(A) = \Gamma_A$, there exist
\( \lambda_1, ..., \lambda_n \geq 0 \) with \( \sum_{k=1}^{n} \lambda_k = 1 \) such that \( z = \sum_{k=1}^{n} \lambda_k x_k \). Let \( y = \sum_{k=1}^{n} \lambda_k x_k \), then \( y \in E \)
and we have \( p_i(z - y) \leq \sum_{k=1}^{n} \lambda_k p_i(x_k - y) < \varepsilon \). Hence \( \Gamma_A \subset B \) and \( B \) is \( H \)-convex. This shows that each locally convex topological vector space must be a l.c.-space with \( \Gamma_{\{x\}} = \{x\} \). Clearly, the inverse is not true. Hence the notion of l.c.-space introduced by Horvath\(^{11}\) is a true generalization of locally convex topological vector spaces.

The following result is Lemma 2 of Tarafdar\(^{18}\).

**Lemma 2.1** — The product of any number of \( H \)-space is an \( H \)-space and the product of \( H \)-convex subsets is \( H \)-convex.

By the definition of l.c.-space and Lemma 2.1, we have that the product of any number of l.c.-spaces is an l.c.-space.

The following result is a consequence of Lemma 2 of Ding-Tan\(^{7}\).

**Lemma 2.2** — Let \( (X, \{\Gamma_A\}) \) be an \( H \)-space and \( \{R_i\}_{i=0}^{n} \) be a family of closed subsets of \( X \). Suppose that there exists a subset \( \{x_0, ..., x_n\} \) of \( X \) such that for each nonempty subset \( J \) of \( \{0, ..., n\} \), \( \Gamma_{\{x_j \mid j \in J\}} \subset \bigcup_{j \in J} R_j \). Then \( \bigcap_{i=0}^{n} R_i \neq \emptyset \).

The following result is Theorem 2 of Horvath\(^{11}\).

**Lemma 2.3** — Let \( X \) be a paracompact topological space, \( (Y, \{\Gamma_A\}) \) be an \( H \)-space and \( S, T : X \to 2^Y \) be two maps such that

(i) for each \( x \in X \), \( S(x) \subset T(x) \),

(ii) for each \( x \in X \) and for each \( A \in \mathcal{F}(S(x)) \), \( \Gamma_A \subset T(x) \),

(iii) \( X = \bigcup \{\text{int}_X S^{-1}(y) : y \in Y\} \), where \( S^{-1}(y) = \{x \in X : y \in S(x)\} \).

Then \( T \) has a continuous selection \( g : X \to Y \), furthermore if \( X \) is compact there is a finite subset \( Y_0 \) of \( Y \) such that \( g(x) \subset \Gamma_{Y_0} \).

3. FIXED POINT THEOREMS

In this section, we prove some fixed point theorems which will be used, in the next section, to prove the equilibrium existence theorems of abstract economies.

**Theorem 3.1** — Let \( X \) be a nonempty \( H \)-convex subset of an l.c.-space \( (Y, \{\Gamma_A\}) \). \( F : X \to 2^Y \) be a lower semicontinuous map such that for each \( x \in X \), \( F(x) \) is nonempty \( H \)-convex in \( Y \). Suppose that there exists a precompact subset \( K \) of \( X \) such that for each \( x \in X \), \( F(x) \cap K \neq \emptyset \). Then for each entourage \( V \in (V_{\delta i})_{i \in I} \), there exists a point \( y_V \in X \) such that

\[ F(y_V) \cap V(y_V) \neq \emptyset \]

**PROOF:** For an arbitrary fixed \( V \in (V_{\delta i})_{i \in I} \), without loss of generality we may assume that \( V \) is open. Define a map \( R : X \to 2^X \) by
\[ R(x) = \{ y \in X : F(y) \cap V(x) = \emptyset \} \text{ for each } x \in X. \]

Since \( V(x) \) is open in \( Y \) and \( F \) is lower semicontinuous, \( R(x) \) is closed in \( X \). For the precompact subset \( K \) of \( X \), we can find a finite subset \( \{ x_0, \ldots, x_n \} \) of \( K \) such that

\[ K \subset \bigcup_{i=0}^n V(x_i). \]

It follows that for each \( x \in X \), \( F(x) \cap \bigcup_{i=0}^n V(x_i) \neq \emptyset \) and hence \( \bigcap_{i=0}^n R(x_i) = \emptyset \).

Note that \( (X, \{ \Gamma_A \}) \) is also an \( H \)-space. By Lemma 2.2, there exists a nonempty subset \( \{ z_0, \ldots, z_k \} \) of \( \{ x_0, \ldots, x_n \} \), \( 0 \leq k \leq n \) and \( y_V \in \Gamma_{\{z_0, \ldots, z_k\}} \) such that \( y_V \not\in \bigcup_{j=0}^k R(z_j) \) so that

\[ F(y_V) \cap V(z_j) \neq \emptyset \text{ for each } j = 0, \ldots, k; \]

that is

\[ z_j \in \{ y \in X : F(y_V) \cap V(y) \neq \emptyset \} \text{ for each } j = 0, \ldots, k. \]

Since \( F(y_V) \) is \( H \)-convex and \( (Y, \{ \Gamma_A \}) \) is an l.c.-space, the set \( \{ y \in X : F(y_V) \cap V(y) \neq \emptyset \} \) is \( H \)-convex. It follows that

\[ y_V \in \Gamma_{\{z_0, \ldots, z_k\}} \subseteq \{ y \in X : F(y_V) \cap V(y) \neq \emptyset \} \]

and hence \( F(y_V) \cap V(y_V) \neq \emptyset \). This completes the proof.

Theorem 3.1 is an almost fixed point theorem for lower semicontinuous maps which generalizes Theorem 7 of Ky Fan\(^9\) to l.c.-spaces.

**Theorem 3.2** — Let \( (X, \{ \Gamma_A \}) \) be a Hausdorff l.c.-space such that for each \( x \in X \), \( \{ x \} = \Gamma_{\{x\}} \); and \( f : X \to X \) be a continuous map such that \( f(X) \) is precompact. Then \( f \) has a fixed point in \( X \).

**Proof**: Since for each \( x \in X \), \( \{ x \} = \Gamma_{\{x\}} \), therefore for each \( x \in X \), \( f(x) \) is \( H \)-convex. By applying Theorem 3.1 with a single valued map, we have that for each entourage \( V \in \mathcal{V} \), there exists \( y_V \in X \) such that \( (y_V, f(y_V)) \subseteq V \). From the compactness of \( f(X) \) it follows that \( f \) must have a fixed point in \( X \).

Theorem 3.2 generalizes Corollary 4.4 of Horvath\(^{11}\) in the following \( X \) need not be a complete metric space. Theorem 3.2 also improves and generalizes Corollary 6 of Ky Fan\(^9\) and Tychonoff's fixed point theorem\(^{24}\) to noncompact and nonconvex setting.

**Theorem 3.3** — Let \( I \) be a finite or infinite index set. For each \( i \in I \), let \( (X_i, \{ \Gamma_{A_i} \}) \) be a Hausdorff l.c.-space with \( \{ x_i \} = \Gamma_{\{x_i\}} \) for each \( x_i \in X_i \) such that
Equilibrium Existence Theorems

$X = \prod X_i$ is paracompact. For each $i \in I$, let $D_i$ be a nonempty compact subset of $X_i$ and $S_i, T_i : X \to 2^{D_i}$ be such that

(i) for each $x \in X$, $S_i(x) \subset T_i(x)$,

(ii) for each $x \in X$ and for each $A_i \in \mathcal{F}(S_i(x))$, $T_i(x) \subset T_i(x)$,

(iii) $X = \bigcup \{ \text{int}_X S_i^{-1}(y_i) : y_i \in D_i \}$.

Then there exists a point $\hat{x} \in D = \prod D_i$ such that $\hat{x} \in T(\hat{x}) = \prod T_i(\hat{x})$, that is, $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$, where $\hat{x}_i$ is the projection of $\hat{x}$ onto $X_i$ for each $i \in I$.

**Proof:** For each $i \in I$, by the conditions (i), (ii), (iii) and Lemma 2.3, there exists a continuous map $g_i : X \to D_i$ such that $g_i(x) \in T_i(x)$ for each $x \in X$. Define the maps $f : X \to D$ and $T : X \to 2^D$ by

$$f(x) = \prod_{i \in I} g_i(x)$$

and $T(x) = \prod_{i \in I} T_i(x)$ for each $x \in X$.

Then $f : X \to X$ is continuous, $\overline{f(X)} \subset D$ is compact and $f(x) \in T(x)$ for each $x \in X$. It follows from Lemma 2.1 and the assumption that $X$ is a Hausdorff paracompact l.c.-space. By applying Theorem 3.2, there exists a point $\hat{x} \in D$ such that $\hat{x} = f(\hat{x}) \in T(\hat{x})$; this completes our proof.

As a consequence of Theorem 3.3, we have the following result.

**Theorem 3.4** — Let $I$ be a finite or infinite index set. For each $i \in I$, let $(X_i, \{ \Gamma_{A_i}^i \})$ be a compact l.c.-space with $\{ x_i \} = \Gamma_i^i \{ x_i \}$ for each $x_i \in X_i$ and $S_i, T_i : X = \prod X_i \to X_i$ be such that

(i) for each $x \in X$, $S_i(x) \subset T_i(x)$,

(ii) for each $x \in X$ and for each $A_i \in \mathcal{F} (S_i(x))$, $\Gamma_{A_i}^i \subset T_i(x)$,

(iii) $X = \bigcup \{ \text{int}_X S_i^{-1}(y_i) : y_i \in X_i \}$.

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x}) = \prod_{i \in I} T_i(\hat{x})$, that is, $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.

**Proof:** The conclusion holds from Theorem 3.3 with $X_i = D_i$ for each $i \in I$.

By putting $0_{x_i} = \text{int}_X S_i^{-1}(y_i)$ for each $i \in I$ and $y_i \in X_i$ and letting $T_i(x)$ is $H$-convex for each $i \in I$ and $x \in X$ in Theorem 3.4, it is easy to see that Theorem 3.4 is an improved version of Theorem 2.1 of Tarafdar'9 in l.c.-spaces.

4. Equilibrium Existence Theorems

In this section, we apply our fixed point theorem to prove the equilibrium existence theorems of abstract economies.

We first give some definitions in equilibrium theory. Let the set $I$ of agents be any (possibly uncountable) set. An abstract economy $E = (X, P, Q, B)_{i \in I}$ is defined
as a family of ordered quadruples \((X_i, P_i, Q_i, B_i)\) where \(P_i : X = \prod_{i \in I} X_i \to 2^{X_i}\) is a preference correspondence, \(Q_i, B_i : X \to 2^{X_i}\) are the constraint correspondences and \(X_i\) is the strategy set of \(i\)th agent for each \(i \in I\). \(\Gamma = (X_i, P_i)_{i \in I}\) will be called a qualitative game. An equilibrium for \(\mathcal{E}\) is a point \(\hat{x} \in X = \prod_{i \in I} X_i\) such that for each \(i \in I\), \(\hat{x}_i \in B_i(\hat{x})\) and \(Q_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset\). When \(Q_i = B_i\) for each \(i \in I\), our definitions of an abstract economy and an equilibrium coincide with the standard definitions, for example in Borglin-Keiding\(^2\) or in Yannelis-Prabhakar\(^2\). A point \(\hat{x} \in X\) is called an equilibrium of the qualitative game \(\Gamma = (X_i, P_i)_{i \in I}\) if \(P_i(\hat{x}) = \emptyset\) for each \(i \in I\).

**Theorem 4.1** — Let \(\mathcal{E} = (X_i, P_i, Q_i, B_i)_{i \in I}\) be a abstract economy such that for each \(i \in I\),

1. \((X_i, \{A_i\})\) is a Hausdroff \(l.c.-\)space with \(\{x_i\} = \Gamma_{x_i}\) for each \(x_i \in X_i\), \(D_i\) is a nonempty compact subset of \(X_i\) and \(X = \prod_{i \in I} X_i\) is para-compact,

2. for each \(x \in X\), \(Q_i(x) \subset B_i(x) \subset D_i\),

3. for each \(x \in G_i = \{x \in X : P_i(x) \cap Q_i(x) = \emptyset\}\), \(A_i \in \mathcal{F}(P_i(x) \cap Q_i(x))\) implies \(\Gamma_{A_i} \subset P_i(x) \cap B_i(x)\) for each \(x \in X \setminus G_i\), \(A_i \in (Q_i(x))\) implies \(\Gamma_{A_i} \subset B_i(x)\),

4. \(X = \bigcup \{\text{int}_x [P_i^{-1}(y_i) \cup (X \setminus G_i)] \cap Q_i^{-1}(y_i) : y_i \in D_i\}\),

5. for each \(x \in X\), \(x_i \notin P_i(x)\).

Then \(\mathcal{E}\) has an equilibrium.

**PROOF:** For each \(i \in I\), define the maps \(S_i, T_i : X \to 2^{D_i}\) by

\[
S_i(x) = \begin{cases} 
P_i(x) \cap Q_i(x), & \text{if } x \in G_i \\
Q_i(x), & \text{if } x \in X \setminus G_i
\end{cases}
\]

\[
T_i(x) = \begin{cases} 
P_i(x) \cap B_i(x), & \text{if } x \in G_i \\
B_i(x), & \text{if } x \in X \setminus G_i
\end{cases}
\]

Then we have the following properties:

(a) for each \(x \in X\), \(S_i(x) \subset T_i(x) \subset D_i\) by (2),

(b) for each \(x \in X\) and for each \(A_i \in \mathcal{F}(S_i(x))\), \(\Gamma_{A_i} \subset T_i(x)\) by (3),

(c) for each \(y_i \in D_i\), we have

\[
S_i^{-1}(y_i) = [P_i^{-1}(y_i) \cap Q_i^{-1}(y_i) \cap G_i] \cup [Q_i^{-1}(y_i) \cap (X \setminus G_i)]
\]

\[
= [P_i^{-1}(y_i) \cup (X \setminus G_i)] \cap Q_i^{-1}(y_i)
\]

and hence \(X = \bigcup \{\text{int}_x S_i^{-1}(y_i) : y_i \in D_i\}\) by (4).
By Theorem 3.3, there exists a point $\hat{x} \in D$ such that $\hat{x} \in T_i(\hat{x})$ for each $i \in I$. It follows from (5) and the definition of $T$ that for each $i \in I$, $\hat{x} \in B_i(\hat{x})$ and $P_i(\hat{x}) \cap Q_i(\hat{x}) = \emptyset$. This completes the proof.

In the following, we shall need the notion of $H$-convex hull introduced by Taraardar

Let $K$ be a nonempty subset of an $H$-space $(X, \{\Gamma_A\})$, we define the $H$-convex hull of $K$, denoted by $H$-$\text{co}(K)$ as

$$H$-$\text{co}(K) = \bigcap \{D \subseteq X : D \text{ is } H\text{-convex and } K \subseteq D\}.$$

Clearly, $H$-$\text{co}(K)$ is the smallest $H$-convex subset containing $K$. By Lemma 1 of Taraardar

$$H$-$\text{co}(K) = \bigcup \{H$-$\text{co}(A) : A \in \mathcal{F}(K)\}.$$

Thus, if $\{\Gamma_A\}$ satisfies that $\{x\} = \Gamma_{\{x\}}$ for each $x \in X$, we must have $K \subseteq H$-$\text{co}(K)$ for each subset $K$ of $X$.

**Theorem 4.2** — Let $\mathcal{E} = (X_i, P_i, Q_i, B_i)_{i \in I}$ be an abstract economy such that for each $i \in I$,

1. $(X_i, \{\Gamma_{A_i}\})$ is a Hausdorff l.c.-space with $\{x_i\} = \Gamma_{\{x_i\}}$ for each $x_i \in X_i$, $D_i$ is a nonempty compact subset of $X_i$ and $X = \prod_{i \in I} X_i$ is paracompact,

2. for each $x \in X$, $H$-$\text{co}(Q_i(x)) \subseteq B_i(x) \subseteq D_i$,

3. $X = \bigcup \{\text{int}_X [(P_i^{-1}(y)) \cup (X \setminus G_i)] \cap Q_i^{-1}(y) : y \in D_i\}$, where

$$G_i = \{x \in X : P_i(x) \cap Q_i(x) \neq \emptyset\},$$

4. for each $x \in X$, $x \notin H$-$\text{co}(P_i(x))$.

Then $\mathcal{E}$ has an equilibrium.

**Proof** : For each $i \in I$, define the maps $S_i, T_i : X \to 2^{D_i}$ by

$$S_i(x) = \begin{cases} H$-$\text{co}(P_i(x)) \cap Q_i(x), & \text{if } x \in G_i, \\ Q_i(x), & \text{if } x \in X \setminus G_i, \end{cases}$$

$$T_i(x) = \begin{cases} H$-$\text{co}(P_i(x)) \cap B_i(x), & \text{if } x \in G_i, \\ B_i(x), & \text{if } x \in X \setminus G_i, \end{cases}$$

Then we have the following properties:

(a) for each $x \in X$, $S_i(x) \subseteq T_i(x) \subseteq D_i$ by (2) and $Q_i(x) \subseteq H$-$\text{co}(Q_i(x))$,

(b) for each $x \in X$ and for each $A_i \in \mathcal{F}(S_i(x))$, if $x \in G_i$, then

$$A_i \in \mathcal{F}(H$-$\text{co}(P_i(x)) \cap Q_i(x)) \subseteq \mathcal{F}(H$-$\text{co}(P_i(x)) \cap H$-$\text{co}(Q_i(x)))$$.
Since $H$-co $(P_i(x)) \cap H$-co$(Q_i(x))$ is $H$-convex, we have
\[ \Gamma_{A_i}^i \subset H$-co $(P_i(x)) \cap H$-co $(Q_i(x)) \subset H$-co $(P_i(x)) \cap B_i(x) \subset T_i(x) \] by (2); if $x \in X \setminus G_i$, then $A_i \in \mathcal{F}(Q_i(x)) \subset \mathcal{F}(H$-co$(Q_i(x)))$ so that $\Gamma_{A_i}^i \subset H$-co$(Q_x(x)) \subset B_i(x) \subset T_i(x)$ by (2). Hence for each $x \in X$ and for each $A_i \in \mathcal{F}(S_i(x))$, $\Gamma_{A_i}^i \subset T_i(x)$.

(c) Since $P_i(x) \subset H$-co $(P_i(x))$ for each $x \in X$, for each $y_i \in D_i$, we have
\[ S_i^1(y_i) = [(H$-co$ P_i)^{-1}(y_i) \cap Q_i^1(y_i) \cap G_i] \cup [Q_i^1(y_i) \cap (X \setminus G_i)] \]
\[ \supset [(P_i^{-1}(y_i) \cap Q_i^1(y_i) \cap G_i] \cup [Q_i^1(y_i) \cap (X \setminus G_i)] \]
\[ = [P_i^{-1}(y_i) \cup (X \setminus G_i) \cap Q_i^1(y_i)]. \]

By (3), we have $X = \bigcup \{ \text{int}_x S_i^1(y_i) : y_i \in D_i \}$.

By Theorem 3.3, there exists a point $\hat{x} \in D$ such that $\hat{x} \in T_i(\hat{x})$ for each $i \in I$. It follows from the definition of $T$ and (4) that for each $i \in I$, $\hat{x} \in B_i(\hat{x})$ and $P_i(\hat{x}) \cap Q_i(\hat{x}) = \emptyset$. This completes our proof.

Theorem 4.2 generalizes Theorem 5 of Ding-Kim-Tan$^4$ to l.c.-spaces. As a direct consequences of Theorem 4.2, we have the following result.

**Corollary 4.1** — Let $\mathcal{E} = (X_i, P_i, Q_i, B_i)_{i \in I}$ be an abstract economy such that for each $i \in I$,

1. $(X_i, \{ \Gamma_i^i \})$ is a Hausdorff compact l.c.-space with $\{X_i \} = \Gamma_i^i$;
2. for each $x \in X = \prod_{i \in I} X_i, \ H$-co $(Q_i(x)) \subset B_i(x)$,
3. $X = \bigcup \{ \text{int}_x [P_i^{-1}(y_i) \cup (X \setminus G_i)] \cap Q_i^1(y_i) : y_i \in X_i \}$,
4. for each $x \in X, \ x_i \notin H$-co $(P_i(x))$.

Then $\mathcal{E}$ has an equilibrium.

**PROOF** : The conclusion holds from Theorem 4.2 with $D_i = X_i$ for each $i \in I$.

It is easy to see that Corollary 4.1 is an improved version of Theorem 3.1 of Tarafdar$^9$.

**Theorem 4.3** — Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that for each $i \in I$,

1. $(X_i, \{ \Gamma_i^i \})$ is a Hausdorff compact l.c. space with $\{X_i \} = \Gamma_i^i$ for each $x_i \in X_i$,
2. $X = \bigcup \{ \text{int}_x [P_i^{-1}(y_i) \cup (X \setminus G_i)] \}$ where $G_i = \{ x \in X : P_i(x) = \emptyset \}$,
(3) for each $x \in X$, $\hat{x} \not\in H^{-\text{co}} (P_i (x))$.

Then $\Gamma$ has an equilibrium, that is, there exists a point $\hat{x} \in X$ such that for each $i \in I$, $P_i (\hat{x}) = \phi$.

**Proof:** For each $i \in I$, if we define the correspondences $Q_i = B_i : X \to 2^X$ by $Q_i (x) = B_i (x) = X_i$ for each $x \in X$, then the theorem will follow from Corollary 4.1.

Theorem 4.3 improves Theorem 3.2 of Tarafdar.

**References**