

# POLYNOMIAL SOLUTIONS FOR THE PELL'S EQUATION

A. M. S. RAMASAMY

*Pondicherry University, Centre for Futures Studies  
Pondicherry 605 014*

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Let  $D$  be a given square-free natural number. In this paper some polynomial solutions are given for the Pell's equations  $A^2 - DB^2 = \pm 1$  which are significant from computational point of view.

## 1. INTRODUCTION

Let  $D$  be a given square-free natural number. It is well known that the Pell's equation

$$A^2 - DB^2 = 1 \quad \dots (1)$$

always has an infinite number of integral solutions (see e.g. Nagell<sup>3</sup>). All the solutions of (1) with positive  $A$  and  $B$  are obtained by the formula  $A_r + B_r\sqrt{D} = (a + b\sqrt{D})^r$  where  $r = 1, 2, 3, \dots$ , and  $a + b\sqrt{D}$  is the fundamental solution of (1), i.e.,  $A_1 = a$  and  $B_1 = b$ .

The Pell's equation

$$A^2 - DB^2 = -1 \quad \dots (2)$$

may not always have integral solutions. When this equation is solvable in integers for some given  $D$ , if  $L + M\sqrt{D}$  be its fundamental solution, we have the relation

$$a + b\sqrt{D} = (L + M\sqrt{D})^2 \quad \dots (3)$$

where  $a + b\sqrt{D}$  is the fundamental solution of (1).

For a given value of  $D$ , in order to determine the fundamental solution of (1), or (2) (when it exists), one may use the method of continued fraction expansion of  $\sqrt{D}$ . The penultimate convergent of this expansion provides the fundamental solution. From computational point of view, it would be advantageous to group the values of  $D$  into several classes such that the values of  $D$  in a specific class are associated with a similar pattern of continued fraction expansion. It is found that polynomial

expressions for  $D$  come in handy for this purpose. As a consequence, some polynomial solutions are given in this paper for the Pell's equations  $A^2 - DB^2 = \pm 1$ .

## 2. THE PELL'S EQUATION $A^2 - DB^2 = 1$

*Theorem 1* — Suppose  $D = 4t^2 + 12t + 5$ , where  $t$  is any natural number. Then  $a + b\sqrt{D}$  is the fundamental solutions of (1) where  $a = 4t^3 + 18t^2 + 24t + 9$ ,  $b = 2t^2 + 6t + 4$ .

PROOF : For real  $x$ , let the symbol  $[x]$  denote the greatest integer  $\leq x$ . We have

$$(2t + 2) < \sqrt{D} < (2t + 3)$$

and so  $[\sqrt{D}] = 2t + 2$ . Now, expanding  $\sqrt{D}$  into a continued fraction expansion, we obtain

$$\begin{aligned} \sqrt{D} &= [\sqrt{D}] + \sqrt{4t^2 + 12t + 5} - (2t + 2) \\ &= (2t + 2) + \frac{1}{\frac{\sqrt{4t^2 + 12t + 5} + (2t + 2)}{4t + 1}} \\ &= (2t + 2) + \frac{1}{\frac{1 + 1}{\frac{\sqrt{4t^2 + 12t + 5} + (2t - 1)}{4}}} \\ &= (2t + 2) + \frac{1}{\frac{1 + 1}{\frac{t + 1}{\frac{\sqrt{4t^2 + 12t + 5} + (2t + 1)}{2t + 1}}}} \\ &= \dots = \left| 2t + 2, 1, t, 2, t, 1, 4t + 4 \right|. \end{aligned}$$

The penultimate convergent is

$$(2t + 2) + \frac{1}{1 + \frac{1}{t + \frac{1}{2 + \frac{1}{t + \frac{1}{1}}}}} = \frac{4t^3 + 18t^2 + 24t + 9}{2t^2 + 6t + 4}.$$

Thus we get the relation

$$(4t^3 + 18t^2 + 24t + 9)^2 - (4t^2 + 12t + 5)(2t^2 + 6t + 4)^2 = 1.$$

*Theorem 2* — Suppose  $D = m^2 t^4 + 2mnt^3 + n^2 t^2 + mt + n$  where  $m, n, t$  are any natural numbers with  $mn \neq 0$ . Then  $a + b\sqrt{D}$  is the fundamental solution of (1) where  $a = 2mt^3 + 2nt^2 + 1$ ,  $b = 2t$ .

PROOF :  $[\sqrt{D}] = mt^2 + nt$  and the continued fraction expansion of  $\sqrt{D}$  is  $\left| mt^2 + nt, 2t, \frac{2mt^3 + 2nt^2 + 1}{2t} \right|$ . The penultimate convergent is  $\frac{2mt^3 + 2nt^2 + 1}{2t}$ .

*Theorem 3* — Suppose  $D = m^2 t^4 + 2mnt^3 + n^2 t^2 + 2mt + 2n$  where  $m, n, t$  are any

natural numbers with  $mn \neq 0$ . Then  $a + b\sqrt{D}$  is the fundamental solution of (1) where  $a = mt^3 + nt^2 + 1$ ,  $b = t$ .

PROOF :  $[\sqrt{D}] = mt^2 + nt$  and the continued fraction expansion of  $\sqrt{D}$  is  $\left| \frac{mt^3 + nt^2 + 1}{t} \right|$ . The penultimate convergent is  $\frac{mt^3 + nt^2 + 1}{t}$ .

3. THE PELL'S EQUATION  $A^2 - DB^2 = -1$

*Theorem 4* — If  $D = 4t^2 + 4t + 5$  where  $t$  is any natural number, then  $L + M\sqrt{D}$  is the fundamental solution of (2) where  $L = 4t^3 + 6t^2 + 6t + 2$ ,  $M = 2t^2 + 2t + 1$ .

PROOF :  $[\sqrt{D}] = 2t + 1$  and the continued fraction expansion of  $\sqrt{D}$  is  $\left| 2t + 1, t, 1, 1, t, 4t + 2 \right|$ . The penultimate convergent is

$$(2t + 1) + \frac{1}{t+} \frac{1}{1+} \frac{1}{1+} \frac{1}{t} = \frac{4t^3 + 6t^2 + 6t + 2}{2t^2 + 2t + 1}$$

Thus we get the relation

$$(4t^3 + 6t^2 + 6t + 2)^2 - (4t^2 + 4t + 5)(2t^2 + 2t + 1)^2 = -1.$$

*Theorem 5* — If  $D = 4t^4 + 4t + 2$ , where  $t$  is any natural number  $\geq 2$ , then  $L + M\sqrt{D}$  is the fundamental solution of (2) where  $L = 4t^4 - 4t^3 + 2t^2 + 2t - 1$ ,  $M = 2t^2 - 2t + 1$ .

PROOF :  $[\sqrt{D}] = 2t^2$  and the continued fraction expansion of  $\sqrt{D}$  is  $\left| 2t^2, t - 1, 1, 1, t - 1, 4t^2 \right|$ . The penultimate convergent is  $\frac{4t^4 - 4t^3 + 2t^2 + 2t - 1}{2t^2 - 2t + 1}$ .

*Theorem 6* — If  $D = 16t^4 + 8t^3 + 9t^2 + 6t + 2$  where  $t$  is any natural number, then  $L + M\sqrt{D}$  is the fundamental solution of (2) where  $L = 16t^4 + 4t^3 + 8t^2 + 3t + 1$ ,  $m = 4t^2 + 1$ .

PROOF :  $[\sqrt{D}] = 4t^2 + t + 1$  and the continued fraction expansion of  $\sqrt{D}$  is  $\left| 4t^2 + t + 1, 2t, 2t, 8t_2 + 2t + 2 \right|$ . The penultimate convergent is  $\frac{16t^4 + 4t^3 + 8t^2 + 3t + 1}{4t^2 + 1}$ .

4. THE GENERAL PELL'S EQUATION

With the help of the following theorem, one may generate the solutions of the general Pell's equation

$$U^2 - DV^2 = N \tag{4}$$

where  $N$  is a non-zero integer.

*Theorem 7* — Let  $u, v$  be integers such that  $u^2 - Dv^2 = N$ , where  $N$  is any given non-zero integer. Then

$$A_{(1)} = kv^2 \pm u$$

$$B_{(1)} = v,$$

$$D_{(1)} = k^2 v^2 \pm 2kv + D$$

satisfy  $A_{(1)}^2 - D_{(1)} B_{(1)}^2 = N$ , where  $k$  is any natural number.

**PROOF :** Taking  $A_{(1)}, B_{(1)}$  and  $D_{(1)}$  as above, we have  $A_{(1)}^2 - D_{(1)} B_{(1)}^2 = u^2 - Dv^2 = N$ .

*Corollary 1* — Let  $k, t$  be natural numbers. Then

$$A = kt^2 \pm 1$$

$$B = t$$

$$D = k^2 t^2 \pm 2k$$

satisfy (1).

## 5. THE DIOPHANTINE EQUATIONS $A^2 - DB^4 = \pm 1$

The solutions of the Diophantine equations

$$A^2 - DB^4 = \pm 1 \quad \dots (5)$$

may be generated from the solutions of the Pell's equation

$$u^2 - 2v^2 = \pm 1 \quad \dots (6)$$

as provided by the following.

*Theorem 8* — Let  $u, v$  be integers such that

$$u^2 - 2v^k = \pm 1. \quad \dots (7)$$

Then

$$A = u (v^k \mp 1)$$

$$B = v$$

$$D = u^2 \mp 4$$

satisfy

$$A^2 - DB^{2k} = \pm 1. \quad \dots (8)$$

**PROOF :** With  $A, B$  and  $D$  as above, we obtain

$$A^2 - DB^{2k} = u^2 \mp 2u^2 v^k \pm 4v^{2k}$$

$$= u^2 \mp 2v^k (u^2 - 2v^k)$$

$$= u^2 \mp 2v^k (\pm 1) = u^2 - 2v^k = \pm 1.$$

When  $k = 2$ , (7) reduces to the Pell's equation (6) with  $D = 2$  and hence it possesses an infinite number of solutions for both signs. Thus there are an infinite number of values of  $D$  for which the equation (5) has non-trivial integral solutions, as provided by Theorem 8. However, when  $k \geq 3$ , by Thue's theorem (Mordell<sup>2</sup>), (7) has at most a finite number of integral solutions. For e.g., in the case of  $k = 4$ , Ljunggren<sup>1</sup> proved that the only positive integer solutions of the Diophantine equation

$$u^2 - 2v^4 = -1$$

are  $(u, v) = (1, 1)$  and  $(239, 13)$ . Using the latter solution of this equation in theorem 8, we obtain the relation

$$6826318^2 - 57125 \cdot 13^8 = -1$$

thus providing a solution of the Diophantine equation

$$A^2 - DB^8 = -1$$

with  $D = 57125$ .

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