A SOLUTION OF THE TARRY-ESCOTT PROBLEM OF DEGREE $r$.

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1. The Tarry-Escott problem of degree exactly $r$ and order $q$, is that of finding $q$ sets $A_1, A_2, A_3, \ldots, A_q$ of $s$ integers each, such that

$$\sigma_k(A_i) = \sigma_k(A_j) \text{ when } 1 \leq k \leq r; \ 1 \leq i, j \leq q; \quad \ldots \quad (1)$$

while

$$\sigma_{r+1}(A_i) \neq \sigma_{r+1}(A_j) \text{ unless } i = j; \quad \ldots \quad (2)$$

where $\sigma_k(A_m)$ denotes the sum of the $k$th powers of the members of $A_m$.

When sets $A$ satisfy the conditions set down above, we write

$$[A_1 = A_2 = A_3 = \ldots = A_q]_r. \quad \ldots \quad (3)$$

The least value of $s$ for which such sets exist is denoted by $M_q(r)$. Gleden\textsuperscript{1} has shown that when $r = 1, 2, 3$ or $5$,

$$M_q(r) = r+1 \text{ for all } q. \quad \ldots \quad (4)$$

Prouhet\textsuperscript{2} in 1861, anticipating Lehmer\textsuperscript{3}, showed how $q^{r+1}$ integers in Arithmetical Progression, could be divided into $q$ sets of $q^r$ members each so as to satisfy conditions (1) and (2).

In this note, I show that

$$M_q(r+1) \leq qM_q(r). \quad \ldots \quad (5)$$

In view of (4), we then get

$$M_q(4) \leq 4q, \text{ and } \ldots \quad (6)$$

$$M_q(r) \leq 6q^{r-5}, r > 5. \quad \ldots \quad (7)$$

These results are, of course, far from ideal\textsuperscript{4}.

2. The set of integers obtained by adding a fixed integer $t$ to each of the members of a set $A$ shall be denoted by $A + t$. If $A$ be a set of $s$ integers and $B$ a set of $j$ integers, then $A + B$ shall denote the set of $sj$ integers obtained by adding each member of set $A$ to each member of set $B$.

Finally, if a set $L$ consists of all the members contained in sets $A, B, C$ and $D$, say, then we write

$$L = A, B, C, D. \quad \ldots \quad (8)$$

Throughout this note, small letters denote integers and capital letters denote sets, unless otherwise stated.

3. It is easily shown that (3) implies

$$[A_1 + t = A_2 + t = A_3 + t = \ldots = A_q + t]_r; \quad \ldots \quad (9)$$

and

$$[A_1 + B = A_2 + B = A_3 + B = \ldots = A_q + B]_r. \quad \ldots \quad (10)$$

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We proceed to show that it also implies
\[ [C_1 = C_2 = C_3 = \ldots = C_s]_{r+1} \ldots \ldots \quad (11) \]
where
\[ C_m = \mu_1 + t_{u_1}, \mu_2 + t_{u_2+1}, \mu_3 + t_{u_3+2}, \ldots, \mu_q + t_{u_q+2} - 1; \]
\[ t_u = u - q, \quad u > q; \]
and the \( t \)'s are not all equal.

Proof of (11).

Let \( A_m \) denote the set of integers
\[ \{a_1, a_2, a_3, \ldots, a_m\}; \]
and \( T \) the set of integers
\[ t_1, t_2, t_3, \ldots, t_q. \]

Then, we have
\[
\sigma_k(\mu_j + t) = \sum_{i=1}^{s} (a_i, i + t)^k.
\]

\[
= \sigma_k(\mu_j) + \binom{k}{1} t \sigma_{k-1}(\mu_j) + \binom{k}{2} t^2 \sigma_{k-2}(\mu_j) + \ldots + \binom{k}{k} t^k \sigma_0(\mu_j),
\]
Hence
\[
\sigma_k(C_m) = \sigma_k(\mu_1 + t_{u_1}) + \sigma_k(\mu_2 + t_{u_2+1}) + \ldots + \sigma_k(\mu_q + t_{u_q+2} - 1)
\]
\[
= \sum_{j=1}^{q} \left\{ \sigma_k(\mu_j) + \binom{k}{1} t_{u_j} \sigma_{k-1}(\mu_j) + \ldots + \binom{k}{k} t_{u_j}^k \sigma_0(\mu_j) \right\}.
\]

For \( 1 \leq k \leq r+1 \),
\[
\sigma_k(C_m) = \sigma_k(A_1, A_2, A_3, \ldots, A_q) + \binom{k}{1} \sigma_1(T) \sigma_{k-1}(A_q)
\]
\[
+ \binom{k}{2} \sigma_2(T) \sigma_{k-2}(A_q) + \ldots + \binom{k}{k} s \sigma_k(T),
\]
because
\[ \sigma_i(\mu_j) = \sigma_i(A_q), \quad 1 \leq i \leq r. \]
Therefore \( \sigma_k(C_m) \) is independent of \( m \) when \( 1 \leq k \leq r+1 \).

When, however, \( k = r+2 \),
\[
\sigma_{r+2}(C_m) = \sigma_{r+2}(A_1, A_2, \ldots, A_q) + \binom{r+2}{1} \sum_{j=1}^{q} t_{u_j+2} \sigma_{r+1}(A_j)
\]
\[
+ \binom{r+2}{2} \sigma_2(T) \sigma_{r+1}(A_q) + \binom{r+2}{3} \sigma_3(T) \sigma_{r-1}(A_q)
\]
\[
+ \ldots + \binom{r+2}{r+2} s \sigma_{r+2}(T),
\]
since
\[ \sigma_{r+2}(A_i) \neq \sigma_{r+1}(A_j), \quad i \neq j. \]
Hence \( \sigma_{r+2}(C_m) \) is not independent of \( m \) unless all \( t \)'s are equal.
This proves the result.
From (4) it now follows that solutions of (3) exist for all \( r \).
Moreover,
\[
M_q(r+1) < q M_q(r),
\]
because each of the sets \( C \) has no more than the total number of members in sets \( A \). We say ‘no more than’ because some of the members may be common to all the sets \( C \) and can be cancelled out. Also, it is well known that
\[
M_q(r) \leq r + 1. \quad \ldots \quad \ldots \quad \ldots \quad (12)
\]

**References.**

4. For \( q = 2 \), Wright has shown that
\[
M_q(r) = O(r^4).
\]

I believe that \( M_q(r) \) is independent of \( q \).

If \( \mu_q(r) \) be the least integer among the partitions of which into exactly \( M_q(r) \) non-zero summands, there are \( q \) which considered as sets \( A_1, A_2, A_3, \ldots, A_q \) satisfy conditions (1) and (2), then \( \mu_q(r) \) would be a function of both \( q \) and \( r \).

Thus, \( \mu_q(1) = 2q \).

Also \( \mu_q(2) = 9 \) and \( \mu_q(3) = 18 \).